The Divergence of Pixel Rays with Adjacent Integer Slopes

Joseph E. Fortune

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Abstract

During a naive attempt to develop a ray-based visibility algorithm for a cell-based game, I noticed adjacent rays would diverge after traveling a certain distance leaving a gap in the visibility scan. I became interested in these critical points of diversion and began investigating their properties. The first discovery was that the y-coordinates of the critical points along the rays followed the sequence $n^2 - 1$ for integers $n \geq 2$. Proving this led to other interesting corollaries.

$\vdots$	$\vdots$	$\vdots$	$\iddots$
(0, 2)	(1, 2)	(2, 2)	$\cdots$
(0, 1)	(1, 1)	(2, 1)	$\cdots$
(0, 0)	(1, 0)	(2, 0)	$\cdots$

Figure 1: The coordinate scheme of the pixel array

Preliminaries

The graph on which we will be plotting the approximations of rays is a matrix of discrete cells henceforth referred to as ”pixels”. Departing from the standard in computing, our origin will be at the bottom left side of the graph which will be extended infinitely in both positive directions.

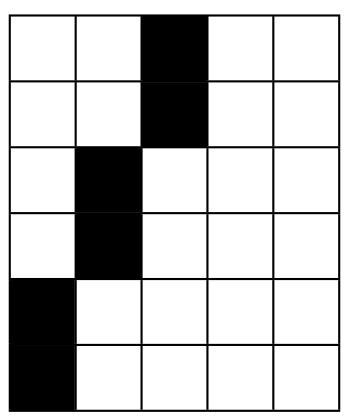

Figure 2: Plotting a ray with a slope of 2

The method of plotting a ray is as follows:

1. From the starting pixel and moving upward, mark the number of pixels that corresponds to the slope (including the starting pixel).
2. Move one unit to the right and one unit up. This will be the starting pixel for the next iteration.
3. Repeat.

Critical Points

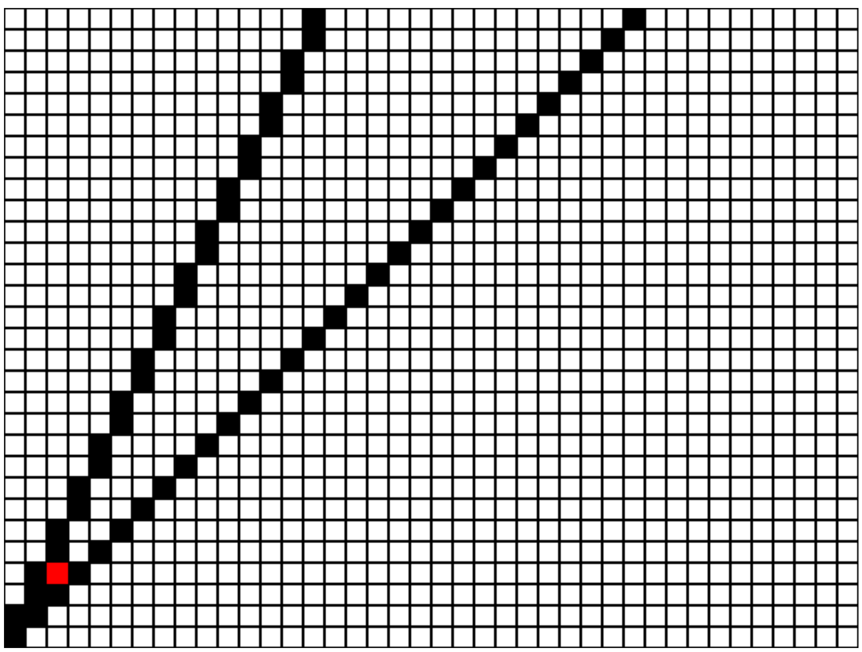

Figure 3: The critical point between two rays

The focus of our investigation is the point of divergence between two rays. Let the first empty pixel between the two arrays be known as the "critical point". It is our purpose to explore the relationship between rays with adjacent slopes and their critical points.

Rays with Adjacent Slopes

Theorem. *If two pixel rays, R_1 and R_2, starting from the origin have a positive integer slopes m_1 and m_2 such that $m_2 - m_1 = 1$, then their critical point is at $(m_2, {m_2}^2 - 1)$.*

Proof. Let R_1 and R_2 be rays with slopes m_1 and m_2, respectively, where $m_2 - m_1 = 1$ and R_1 and R_2 originate from $(0, 0)$.

At $x = 0$, R_1 will have m_1 height while R_2 will have $m_1 + 1$ height.

At $x = 1$, R_1 will have $2m_1$ height while R_2 will have $2(m_1 + 1)$ height.

$$\vdots$$

At $x = k$, R_1 will have km_1 height while R_2 will have $k(m_1 + 1)$ height.

Following these iterations, a vertical gap (critical point) will form between the rays when R_2 reaches a height such that

$$k(m_1 + 1) - m_1 - km_1 = 1$$

$$k = m_1 + 1$$

$$k = m_2$$

Therefore the x-coordinate of the critical point lies at $x = m_2$.

To find the corresponding y-coordinate, we will use $y = mx$ which given the discrete nature of the pixel scheme gives us the smallest y-coordinate plotted in that vertical column for that ray. Since the critical point lies immediately below R_2, we can state that the y-coordinate of the critical point is

$$y_c = y_2 - 1$$

$$y_c = m_2 x_2 - 1$$

As previously discovered, the x-coordinate of the critical point is m_2

$$y_c = (m_2)(m_2) - 1$$

$$y_c = {m_2}^2 - 1$$

Therefore, the critical point is $(m_2, {m_2}^2 - 1)$. ∎