Bezier Curves

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Abstract:

An exploration of the development of Bézier curves and a further study of their properties which suit them to be widely used in computer graphics, along with their origin in Bernstein polynomials and the Weierstrass approximation theorem. In general, the paper follows the chronological order in which the Bézier curves were developed.

Contents

Introduction

The realm of computer graphics is a wonderful playground for the mathematician. The challenges of jumping from the continuous to the discrete and back, and the need for more elegant representations of geometric structures to shave milliseconds off a program cycle has reawaken interest in fields of mathematics that were perhaps once seen as mere fanciful curiosities. Several of these once obscure theorems have been dusted off and pushed to the forefront of daily use. It is both fascinating and encouraging that several mathematicians were exploring curiosities for their own curiosity sake, and stumbled upon an observation that would later form the foundation of tangible constructions in the modern world.

Of course, the mathematics behind computer graphics are too vast for any single book, much less this paper. The scope of this paper will be narrowed to an exploration of Bézier curves, a structure who's history, like so many other applied mathematical structures, began as abstract curiosity. However, unlike many fields of mathematics who's development is harder to trace, as their proofs, theorems, and notation have been refined over the ages, Bézier curves enjoy a periodic development that corresponds to the way in which we derive them. That is to say, we derive them in the same order as they were developed, which gives wonderful insight into their evolution. What would become one of the most efficient and widely used methods of modern curve rendering, all started, as you will see ahead, by a derivation beginning with the number 1.

Section 1 Bernstein Polynomials

Polynomials are useful tools in mathematics as they are simply defined, can mimic other functions, and can be computed quickly. The *Bernstein polynomial* emerged from the field of numerical analysis and is named after Serge Bernstein, a Soviet mathematician who is the first to utilize them in a constructive proof for the Stone-Weierstrass approximation theorem.

Section 1.1 Deriving the Bernstein polynomial.

The Bernstein polynomial of degree n is defined as

$$B_i^n(u) = \binom{n}{i} u^i (1-u)^{n-i}, \quad u \in \mathbf{R}, \quad i = 0, \ldots, n$$

where for mathematical convenience, $B_i^n(u)$ is defined such that $B_i^n(u) = 0$ if $i < 0$ or $i > n$.

[1, pg. 3]. We obtain the Bernstein polynomial by using the binomial expansion formula,

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

to represent the expansion that equates to 1, which gives us the Bernstein polynomial,

$$1 = (u + (1-u))^n = \sum_{i=0}^{n} \binom{n}{i} u^i (1-u)^{n-i}$$

Theorem 1.1. Bernstein polynomials form a *partition of unity* [2, pg. 10].

Proof. A set of functions $f_i(t)$ is said to partition unity if they sum to 1 for all values of t. As we see from the construction of the Bernstein polynomial,

$$\sum_{i=0}^{n} B_i^n(u) = \sum_{i=0}^{n} \binom{n}{i} u^i (1-u)^{n-i} = 1$$

■

A few examples of the first Bernstein polynomials are:

$B_0^0(u) = 1$

$B_0^1(u) = 1 - u$ $\quad$ $B_1^1(u) = u$

$B_0^2(u) = (1-u)^2$ $\quad$ $B_1^2(u) = 2u(1-u)$ $\quad$ $B_2^2(u) = u^2$

$B_0^3(u) = (1-u)^3$ $\quad$ $B_1^3(u) = 3u(1-u)^2$ $\quad$ $B_2^3(u) = 3u^2(1-u)$ $\quad$ $B_3^3(u) = u^3$

[1, pg.3]

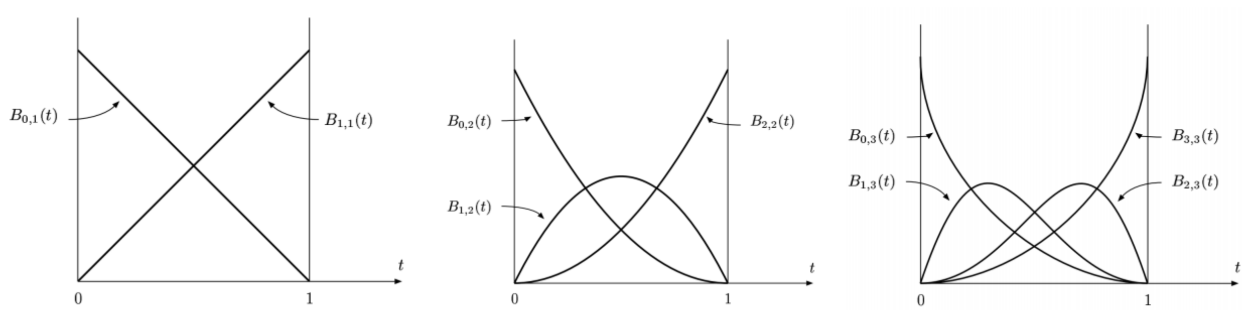

Figure 1.1 Bernstein Basis polynomials of degree 1, 2, and 3 [1, pg.4,5].

Using the following identity, we can construct a recursive definition:

$$\binom{n+1}{i} = \binom{n}{i-1} + \binom{n}{i}$$

Consider the polynomials $B_{i-1}^{n}(u)$ and $B_i^n(u)$. Multiplying them by u and $(1-u)$, respectively, we obtain

$$uB_{i-1}^{n}(u) = u\left[\binom{n}{i-1}u^{i-1}(1-u)^{n+i-1}\right] = \binom{n}{i-1}u^{i}(1-u)^{n+1-i}$$

$$(1-u)B_{i}^{n}(u) = u\left[\binom{n}{i}u^{i}(1-u)^{n+i}\right] = \binom{n}{i}u^{i}(1-u)^{n+1-i}$$

Then adding $uB_{i-1}^{n}(u)$ and $(1-u)B_i^n(u)$ together, we get

$$\begin{aligned} uB_{i-1}^{n}(u) + (1-u)B_{i}^{n}(u) &= \binom{n}{i-1}u^{i}(1-u)^{n+1-i} + \binom{n}{i}u^{i}(1-u)^{n+1-i} \\ &= u^{i}(1-u)^{n+1-i}\left[\binom{n}{i-1} + \binom{n}{i}\right] \\ &= u^{i}(1-u)^{n+1-i}\binom{n+1}{i} \\ &= B_{i}^{n+1} \end{aligned}$$

Thus, we obtain the recursive formula $B_i^{n+1}(u) = uB_{i-1}^{n}(u) + (1-u)B_i^n(u)$ [2, pg. 10].

We can represent the computation of the Bernstein polynomials up to degree n in the triangle below, where polynomials along the horizontal represent $1-u$, and polynomials along the diagonal represent u.

$$\begin{array}{ccccc} 1 = B_0^0 & B_0^1 & B_0^2 & \cdots & B_0^n \\ & B_1^1 & B_1^2 & \cdots & B_1^n \\ & & B_2^2 & \cdots & B_2^n \end{array}$$

$$\ddots \quad \vdots$$

$$B_n^n$$

[2, pg. 11]

Theorem 1.2. Bernstein polynomials are non-negative on the interval [0,1], and are strictly positive on the open interval $(0,1)$ [1, pg. 6].

Proof. To prove this, we will use mathematical induction on the previously defined recursive definition.

Basis case: $B_0^0(u) = 1 \geq 0$

Inductive hypothesis: Assume $B_i^n(u) \geq 0$ for all $i, j < n$ for some n.

By our recursive definition, $B_i^{n+1}(u) = uB_{i-1}^n(u) + (1-u)B_i^n(u)$. Since the right-hand side of the equation consists of all non-negative components (recall, if $i < 0$, then $B_i^n(u) = 0$), we can conclude that $B_i^{n+1}(u) \geq 0$ for $0 \leq u \leq 1$. Therefore, by induction, all Bernstein polynomials are nonnegative for $0 \leq u \leq 1$. Modifying our hypothesis to be the open set $0 < u < 1$, we see that $B_i^n(u)$ is strictly positive [1, pg. 6].

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Other properties include that they are symmetric, $B_i^n(u) = B_i^{n-1}(1-u)$

They have roots at 0 and 1 only, $B_i^n(0) = B_{n-i}^n(1) = \begin{cases} 1 & i = 0 \\ 0 & i > 0 \end{cases}$

Theorem 1.3. Bernstein polynomials are linearly independent [2, pg. 1].

Proof. We will show that if the sum of Bernstein polynomials equal zero, then all its coefficients are zero. Let c_i be the coefficient of $B_i^n(u)$. We have

$$\sum_{i=0}^{n} c_i u^i (1-u)^{n-i} = 0$$

Dividing both sides by $(1-u)^n$, we get

$$\sum_{i=0}^{n} c_i u^i (1-u)^{-i} = 0$$

Let $s_i = u^i(1-u)^{-i}$, then

$$\sum_{i=0}^{n} c_i s_i = 0$$

The linear independence property states that for a basis $B = \{v_1, v_2, \dots, v_n\}$ which is a finite subset of a vector space over a field **F** then for all $a_1, a_2, \dots, a_n \in \mathbf{F}$, if $a_1 v_1 + \cdots + a_n v_n = 0$, then necessarily $a_1 = \cdots = a_n = 0$. From our summation above, we have $c_0 s_0 + \cdots + c_n s_n$, which implies that $c_0 = \cdots = c_n = 0$ and thus establishes linear independence. ∎

[2, pg. 1,2]

Section 1.2 The Weierstrass Approximation Theorem

To conclude our discussion of Bernstein polynomials, we will explore them in their intended design to prove the Weierstrass approximation theorem. The intent of the theorem is to demonstrate that any continuous function can be uniformly approximated over a closed interval with unlimited precision. Note that this differs from the familiar Taylor series as the Weierstrass approximation theorem does not require that the function be differentiable. Also the Weierstrass approximation theorem approximates a function over an interval, rather than a single point.

Theorem 1.4. The Weierstrass Approximation Theorem

If $F(x)$ is any continuous function in the interval $[0,1]$, it is always possible, regardless of how small ε, to determine a polynomial $E_n(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ of degree n high enough such that we have $|F(x) - E_n(x)| < \varepsilon$ for every point in the interval under consideration.

Proof. Consider an event A, whose probability is equal to x. Suppose n experiments are conducted and it is agreed that the player will be payed a sum of $F\left(\frac{i}{n}\right)$, if event A occurs i times. Using the binomial probability formula, we obtain the expected value E_n where

$$E_n = \sum_{i=0}^{n} F\left(\frac{i}{n}\right) \cdot \binom{n}{i} x^i (1-x)^{n-i}$$

Because $F(x)$ is continuous, it is possible to find a number δ such that $|x - x_0| \leq \delta$ which implies that

$$|F(x) - F(x_0)| \leq \frac{\varepsilon}{2}$$

We will denote $\overline{F}(x)$ as the maximum and $\underline{F}(x)$ as the minimum on the interval $(x-\delta, x+\delta)$, so

$$\overline{F}(x) - F(x) < \frac{\varepsilon}{2}, \qquad F(x) - \underline{F}(x) < \frac{\varepsilon}{2}$$

Let p be the probability of the inequality $\left|x - \frac{i}{n}\right| > \delta$ and L the maximum of $|F(x)|$ over the interval $[0,1]$. Then we obtain

$$F(x) \cdot (1-p) - Lp < E_n < F(x) + \left(\overline{F}(x) - F(x)\right) + p(L - \overline{F}(x))$$

So,

$$F(x) - \frac{\varepsilon}{2} - \frac{2L}{4L}\varepsilon < E_n < F(x) + \frac{\varepsilon}{2} + \frac{2L}{4L}\varepsilon$$

Thus, we conclude

$$|F(x) - E_n| < \varepsilon$$

■

Example 1.1

Suppose we wish to approximate the function $\sin(\theta)$ over the interval $0 \leq \theta \leq 2\pi$. Since the Bernstein polynomial is only defined over the interval $[0,1]$, we must shrink our function such that one period is traversed over $[0,1]$. To achieve this, we transform $\sin(\theta)$ into $\sin(2\pi\theta)$. Then our approximating polynomial, we will call $f(x)$, is

$$\sin(t) \approx \sum_{i=0}^{n} \sin\left(\frac{2\pi i}{n}\right) \cdot B_i^n(t)$$

Let $n = 3$. Even at this low degree we already see the graph of $f(x)$ taking familiar shape, and the function is

$$f(x) = \sin(0)(1-t)^3 + 3\sin\left(\frac{2\pi}{3}\right)(1-t)^2 t + 3\sin\left(\frac{4\pi}{3}\right)(1-t)t^2 + \sin\left(\frac{\pi}{2}\right)t^3$$

$$= \frac{3\sqrt{3}}{2}(1-t)^2 t - \frac{3\sqrt{3}}{2}(1-t)t^2 + t^3$$

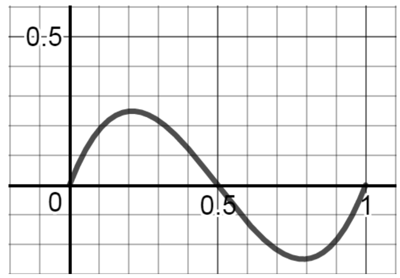

Figure 1.2 Approximation of $\sin(\theta)$ using the Bernstein polynomial of degree 3.

Section 2 Bézier Curves

Bézier curves are used extensively in computer graphics to model smooth curves. Rather than being defined by an equation directly, a Bézier curve is described by a series of control points of which the curve itself is an interpolation. This lends itself to being very simple to perform affine transformations on the curve since it is simply a matter of transforming control points rather than having to manipulate an equation. The curve is named after Pierre Bézier who was the first to utilize the curve in the application of designing automobiles. Paul de Casteljau first developed the curve for evaluating polynomials, but it was Bézier who popularized the curve's use for modeling and design. The Bézier curve is defined as

$$\boldsymbol{B}(t) = \sum_{i=0}^{n} \boldsymbol{b}_i B_i^n(t), \qquad 0 \leq t \leq 1$$

where $\boldsymbol{b}_0, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$ are the $n+1$ control points that describe the curve [3, pg. 141]. Some references on Bézier curves represent the control points as elements of $\mathbb{R}^n$ and parameterize an equation for each dimension, whereas others represent the control points as n-degree vectors. This paper will use the latter for conciseness and because it is the method used in computing applications. Now let's examine some of the lower-degree Bézier curves.

2.1 Linear Bézier curves

Linear Bézier curves are Bézier curves where $n = 1$, which implies that we are using two control points which we will call $\boldsymbol{p}_0$ and $\boldsymbol{p}_1$. Then the Bézier curve is

$$\boldsymbol{B}(t) = \sum_{i=0}^{1} \boldsymbol{b}_i B_i^1(t) = \boldsymbol{p}_0 B_0^1(t) + \boldsymbol{p}_1 B_1^1(t)$$

$$= \boldsymbol{p}_0(1-t) + \boldsymbol{p}_1 t, \quad 0 \le t \le 1$$

The linear Bézier curve is defined on the closed interval [0, 1], such that the starting point of the curve is $\boldsymbol{B}(0) = \boldsymbol{p}_0$ and the ending point is $\boldsymbol{B}(1) = \boldsymbol{p}_1$ [3, pg. 136].

Example 2.1

Let $\boldsymbol{p}_0 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \boldsymbol{p}_1 = \begin{bmatrix} -3 \\ -7 \end{bmatrix}$ be control points. Then the linear Bézier curve is

$$\boldsymbol{B}(t) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}(1-t) + \begin{bmatrix} -3 \\ -7 \end{bmatrix} t$$

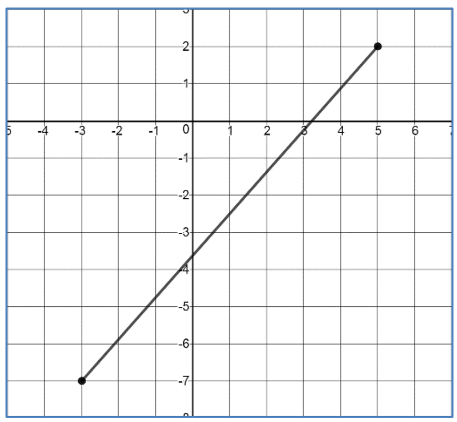

Figure 2.1 Linear Bézier curve with control points (5, 2) and (-3, -7).

2.2 Quadratic Bézier curves

The *Quadratic Bézier curve* is a Bézier curve where $n = 2$ and thus is specified by three control points $\boldsymbol{p}_0$, $\boldsymbol{p}_1$, and $\boldsymbol{p}_2$ and is defined as

$$\boldsymbol{B}(t) = \sum_{i=0}^{2} \boldsymbol{b}_i B_i^2(t) = \boldsymbol{p}_0 B_0^2(t) + \boldsymbol{p}_1 B_1^2(t) + \boldsymbol{p}_2 B_2^2(t)$$

$$= \boldsymbol{p}_0(1-t)^2 + 2\boldsymbol{p}_1(1-t)t + \boldsymbol{p}_2 t^2, \quad 0 \leq t \leq 1$$

[3, pg. 136]

Now for degree $n \geq 2$ we can connect the adjacent points together in what is called the *control polygon*. In this case the control polygon is the triangle $\boldsymbol{p}_0\boldsymbol{p}_1\boldsymbol{p}_2$ [3, pg. 136]. We will explore more about the control polygons ahead.

Example 2.2

Let $\boldsymbol{p}_0 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \boldsymbol{p}_1 = \begin{bmatrix} -3 \\ -7 \end{bmatrix}, \boldsymbol{p}_2 = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$ be control points. Then the quadratic Bézier curve is

$$\boldsymbol{B}(t) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}(1-t)^2 + 2\begin{bmatrix} -3 \\ -7 \end{bmatrix}(1-t)t + \begin{bmatrix} -4 \\ 0 \end{bmatrix}t^2$$

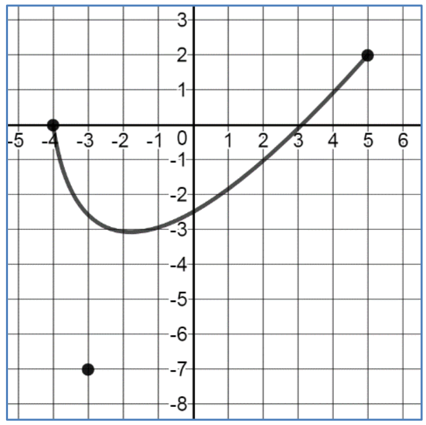

Figure 2.2 Quadratic Bézier curve with control points (5, 2), (-3, -7), and (-4, 0).

2.3 Cubic Bézier curves

For $n = 3$ we will have four control points $\boldsymbol{p}_0$, $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, and $\boldsymbol{p}_3$. The *Cubic Bézier curve* is then

$$\boldsymbol{B}(t) = \sum_{i=0}^{3} \boldsymbol{b}_i B_i^3(t) = \boldsymbol{p}_0 B_0^3(t) + \boldsymbol{p}_1 B_1^3(t) + \boldsymbol{p}_2 B_2^3(t) + \boldsymbol{p}_3 B_3^3(t)$$

$$= \boldsymbol{p}_0(1-t)^3 + 3\boldsymbol{p}_1(1-t)^2 t + 3\boldsymbol{p}_2(1-t)t^2 + \boldsymbol{p}_3 t^3, 0 \le t \le 1$$

Example 2.3

Let $\boldsymbol{p}_0 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \boldsymbol{p}_1 = \begin{bmatrix} -2 \\ 6 \end{bmatrix}, \boldsymbol{p}_2 = \begin{bmatrix} -5 \\ 4 \end{bmatrix}, \boldsymbol{p}_3 = \begin{bmatrix} -6 \\ 0 \end{bmatrix}$ be control points. Then the cubic Bézier curve is

$$\boldsymbol{B}(t) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} (1-t)^3 + 3 \begin{bmatrix} -2 \\ 6 \end{bmatrix} (1-t)^2 t + 3 \begin{bmatrix} -5 \\ 4 \end{bmatrix} (1-t) t^2 + \begin{bmatrix} -6 \\ 0 \end{bmatrix} t^3$$

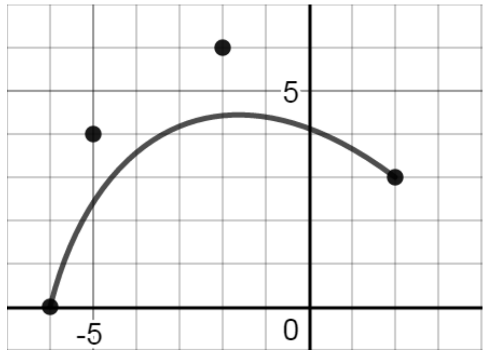

Figure 2.3. Cubic Bézier curve with control points $(2, 3), (-2, 6), (-5, 4)$ and $(-6, 0)$.

Cubic Bézier curves are able to form shapes that quadratic Bézier curves cannot as they have the ability to form loops, cusps, and inflections [3, pg. 136]. We will examine more interesting properties of the Cubic Bézier curve, but first we need to establish how to differentiate a Bézier curve.

Section 3 Derivatives of Bézier curves

To differentiate a Bézier curve, we must first understand the derivative of the underlying Bernstein polynomial.

Theorem 3.1. The derivative of a Bernstein polynomial is

$$\frac{d}{du} B_i^n(u) = n\left(B_{i-1}^{n-1}(u) - B_i^{n-1}(u)\right)$$

[3, pg. 163]

Proof. Taking the derivative of the Bernstein polynomial, we obtain

$$\frac{d}{du}B_i^n(u) = \frac{d}{du}\left[\binom{n}{i}u^i(1-u)^{n-i}\right]$$

$$= \frac{n!}{(n-i)!\,i!}\left[iu^{i-1}(1-u)^{n-i} - u^i(n-i)(1-u)^{n-i-1}\right]$$

$$= \frac{n!}{(n-i)!\,(i-1)!}\cdot u^{i-1}(1-u)^{n-i} - \frac{n!}{(n-i-1)!\,i!}\cdot u^i(1-u)^{n-(i+1)}$$

$$= n\left[\binom{n-1}{i-1}u^{i-1}(1-u)^{n-i} - \binom{n-1}{i}u^i(1-u)^{n-(i+1)}\right]$$

$$= n[B_{i-1}^{n-1}(u) - B_i^{n-1}(u)]$$

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Theorem 3.2. The derivative of a Bézier curve of degree n is

$$\boldsymbol{B}'(t) = \sum_{i=0}^{n-1} n(\boldsymbol{b}_{i+1} - \boldsymbol{b}_i)B_i^{n-1}(t)$$

[3, pg. 163]

Proof. Using the derivative of the Bernstein polynomial $B_i^{n\prime}(t) = n\left(b_{i-1}^{n-1}(u) - b_i^{n-1}(u)\right)$ and the fact that $B_{-1}^{n-1}(t) = B_n^{n-1}(t) = 0$, we get

$$\boldsymbol{B}'(t) = \sum_{i=0}^{n} \boldsymbol{b}_i B_i^{n\prime}(t) = \sum_{i=0}^{n} \boldsymbol{b}_i n\left(B_{i-1}^{n-1}(t) - B_i^{n-1}(t)\right)$$

$$= \sum_{i=0}^{n} n\boldsymbol{b}_i B_{i-1}^{n-1}(t) - \sum_{i=0}^{n} n\boldsymbol{b}_i B_i^{n-1}(t)$$

$$= \sum_{i=0}^{n} n\boldsymbol{b}_i B_{i-1}^{n-1}(t) - \sum_{i=0}^{n-1} n\boldsymbol{b}_i B_i^{n-1}(t)$$

$$= \sum_{i=0}^{n-1} n\boldsymbol{b}_{i+1} B_i^{n-1}(t) - \sum_{i=0}^{n-1} n\boldsymbol{b}_i B_i^{n-1}(t)$$

$$= \sum_{i=0}^{n-1} n(\boldsymbol{b}_{i+1} - \boldsymbol{b}_i) B_i^{n-1}(t)$$

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Corollary 3.1. The rth derivative of a Bézier curve of degree n is

$$\boldsymbol{B}^{(r)}(t) = \sum_{i=0}^{n-r} \boldsymbol{b}_i^{(r)} B_i^{n-r}(t)$$

where

$$\boldsymbol{b}_i^{(r)} = n(n-1)\cdots(n-r+1)\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}\boldsymbol{b}_{i+j}$$

[3, pg. 164]

Now that we are armed with the derivative, we can further explore properties of the cubic Bézier curve. The derivative of the cubic Bézier curve is

$$\boldsymbol{B}'(t) = -3(1-t)^2\boldsymbol{b}_0 + 3(1-4t+3t^2)\boldsymbol{b}_1 + 3t(2-3t)\boldsymbol{b}_2 + 3t^2\boldsymbol{b}_3, \quad 0 \le t \le 1$$

Then $\boldsymbol{B}'(0) = -3(\boldsymbol{b}_1 - \boldsymbol{b}_0)$ which implies that the tangent of $\boldsymbol{B}(t)$ at $t = 0$, or $\boldsymbol{b}_0$, is in the same direction as the vector $\overrightarrow{\boldsymbol{b}_0\boldsymbol{b}_1}$. Likewise, we see that $\boldsymbol{B}'(1) = 3(\boldsymbol{b}_3 - \boldsymbol{b}_2)$. Hence, the final point $\boldsymbol{b}_3$ of $\boldsymbol{B}'(t)$ has the tangent equal to the direction of $\overrightarrow{\boldsymbol{b}_2\boldsymbol{b}_3}$. In both cases, we see that the magnitude of the tangent vector is 3 times the length of the line segment joining the control points. From this, we conclude that Bézier curves exhibit the *endpoint tangent property*. That is, the starting point and starting direction of the curve is dictated by the first two control points, and the end direction and endpoint is dictated by the two final points [3, pg. 138].

Section 4 Properties of Bézier Curves

4.1 Convex Combinations

What is it about the sequence of control points that influence the movement of the curve as we traverse t? To understand the movement, we first need to define *convex combinations*.

Definition 4.1 Convex Combination

Given a set of points $\boldsymbol{p}_0, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_n$, we can form affine combinations of these points by selecting $\alpha_0, \alpha_1, \ldots, \alpha_n$, such that $\alpha_0 + \alpha_1 + \cdots + \alpha_n = 1$ and form the point

$$\boldsymbol{p} = \alpha_0\boldsymbol{p}_0 + \alpha_1\boldsymbol{p}_1 + \cdots + \alpha_n\boldsymbol{p}_n$$

If $0 \le \alpha_i \le 1$, then $\boldsymbol{p}$ is called a *convex combination* of the points $\boldsymbol{p}_0, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_n$ [4, pg. 6].

Theorem 4.1. A point $\boldsymbol{B}(t)$ along a Bézier curve is *a convex combination* of the control points.

Proof. Consider the Bézier curve formula

$$\boldsymbol{B}(t) = \sum_{i=0}^{n} \boldsymbol{b}_i B_i^n(t), \qquad 0 \leq t \leq 1$$

Now substitute $\alpha_i = B_i^n(t)$ and $\boldsymbol{p}_i = \boldsymbol{b}_i$. Then $\boldsymbol{B}(t)$ can be written in the form

$\boldsymbol{B}(t) = \alpha_0 \boldsymbol{p}_0 + \alpha_1 \boldsymbol{p}_1 + \cdots + \alpha_n \boldsymbol{p}_n$ where $\alpha_0 + \alpha_1 + \cdots + \alpha_n = 1$ as the Bernstein polynomials form a partition of unity. Thus $\boldsymbol{B}(t)$ is a convex combination of the points $\boldsymbol{b}_i$, $0 \leq i \leq n$.

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The Bézier curve being a convex combination gives us insight into why the curve behaves the way that it does as it traverses t. A point of a convex combination is simply a weighted average of its component points.

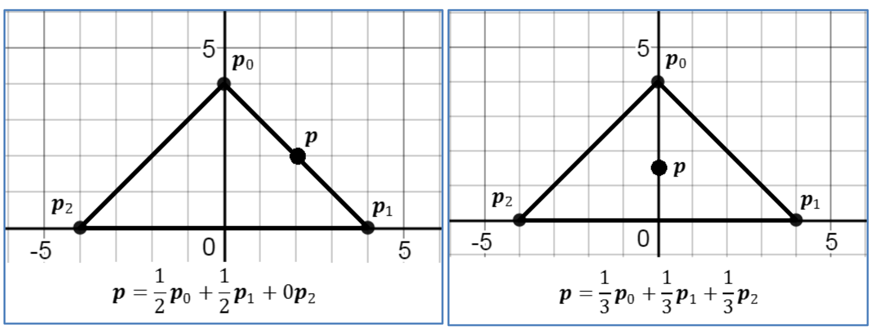

Figure 4.1 and 4.2. Plotting $\boldsymbol{p}$ as a convex combination of $\boldsymbol{p}_0$, $\boldsymbol{p}_1$, and $\boldsymbol{p}_2$.

Thus, a single point, $\boldsymbol{B}(t)$, is merely a weighted average of all its control points. Table 4.1 shows the weights of the individual terms of the cubic Bézier curve over t. We see that each term has an interval of t where it gives its control point the greatest weight, thus giving that control point the greatest influence over the curve during that interval.

t	$(1-t)^3$	$3(1-t)^2t$	$3(1-t)^2$	t^3
0	1	0	0	0
0.1	0.729	0.243	0.027	0.001
0.2	0.512	0.384	0.096	0.008
0.3	0.343	0.441	0.189	0.027
0.4	0.216	0.432	0.288	0.064
0.5	0.125	0.375	0.375	0.125
0.6	0.064	0.288	0.432	0.216
0.7	0.027	0.189	0.441	0.343
0.8	0.008	0.096	0.384	0.512
0.9	0.001	0.027	0.243	0.729
1	0	0	0	1

Table 4.1. Weights of individual terms of the cubic Bézier curve over t (greatest weight highlighted)

Section 4.2 Convex Hull Property

The *convex hull property* is an important property of Bézier curves that is used to derive algorithms for graphical rendering, and for finding the intersection of two Bézier curves.

Definition 4.2 Convex Hull

Given a set of points $X = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_n\}$, the *convex hull* of X, denoted as $\mathrm{CH}\{X\}$, is defined as the set of points

$$\mathrm{CH}\{X\} = \left\{a_0\boldsymbol{x}_0 + a_1\boldsymbol{x}_1 + \cdots + a_n\boldsymbol{x}_n \mid \sum_{i=0}^{n} a_i = 1, a_i \geq 0\right\}$$

[3, pg. 146]

The convex hull can be visualized by picturing points as pegs on a board, and then taking a rubber band and wrapping it around the outermost pegs such that every peg in within the perimeter of the rubber band.

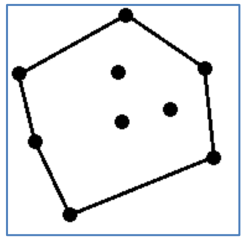

Figure 4.3. The convex hull of a set of points.

Theorem 4.2. Convex Hull Property

For all t, $0 \leq t \leq 1$, $\boldsymbol{B}(t) \in \mathtt{CH}\{\boldsymbol{b}_0, \boldsymbol{b}_1, \dots, \boldsymbol{b}_n\}$. That is, every point of a Bézier curve lies inside the convex hull of its control points. The convex hull of the control points is referred to as the convex hull of the Bézier curve [3, pg. 147]. The proof follows directly from the Bézier curve's property of being a partition of unity and is functionally identical to the proof in Theorem 4.1.

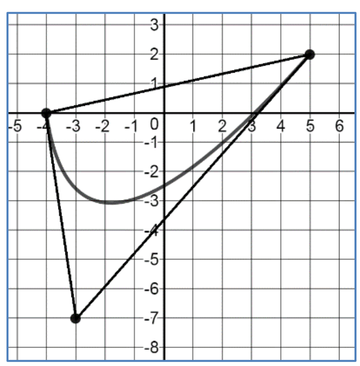

Figure 4.4. The convex hull of a quadratic Bézier curve.

Section 4.3. Invariance Under Affine Transformations

One of the great advantages of using Bézier curves is the simplicity of applying transformations, as a transformation of the curve is simply achieved by transforming its control points. It is important that these transformations preserve certain geometric properties in order to take advantage of this benefit.

Theorem 4.3. Bézier curves are invariant under affine transformations.

Let T bean affine transformation. Then

$$\mathrm{T}\left(\sum_{i=0}^{n} \boldsymbol{b}_i B_i^n(t)\right) = \sum_{i=0}^{n} \mathrm{T}(\boldsymbol{b}_i) B_i^n(t)$$

[3, pg. 147]

Proof. Let T be an affine transformation given by $(x', y') = (ax + by + c, dx + ey + f)$ and let a Bézier curve $\boldsymbol{B}(t)$ of degree n have control points $\boldsymbol{b}_i$ for $0 \leq i \leq n$. Then

$$\boldsymbol{B}(t) = \big(x(t), y(t)\big) = \left(\sum_{i=0}^{n} p_i B_i^n(t), \sum_{i=0}^{n} q_i B_i^n(t)\right)$$

Then apply the transformation and get

$$\mathrm{T}\big(\boldsymbol{B}(t)\big) = \left(a \sum_{i=0}^{n} p_i B_i^n(t) + b \sum_{i=0}^{n} q_i B_i^n(t) + c, d \sum_{i=0}^{n} p_i B_i^n(t) + e \sum_{i=0}^{n} q_i B_i^n(t) + f\right)$$

Because of the partition of unity property, that is,

$$\sum_{i=0}^{n} B_i^n(t) = 1,$$

$$\mathrm{T}(\boldsymbol{B}(t)) = \left(a\sum_{i=0}^{n} p_i B_i^n(t) + b\sum_{i=0}^{n} q_i B_i^n(t) + c\sum_{i=0}^{n} B_i^n(t), d\sum_{i=0}^{n} p_i B_i^n(t) + e\sum_{i=0}^{n} q_i B_i^n(t) + f\sum_{i=0}^{n} B_i^n(t)\right)$$

$$= \left(\sum_{i=0}^{n}(ap_i + bq_i + c)\, B_i^n(t), \sum_{i=0}^{n}(dp_i + eq_i + f)\, B_i^n(t)\right)$$

$$= \sum_{i=0}^{n}(ap_i + bq_i + c, dp_i + eq_i + f)\, B_i^n(t)$$

$$= \sum_{i=0}^{n} \mathrm{T}(\boldsymbol{b}_i) B_i^n(t)$$

■

Section 4.4 Other Properties

The remaining properties are generalized results of what we have already explored and were implicitly mentioned, but are stated here explicitly for completeness.

Endpoint Interpolation Property: $\boldsymbol{B}(0) = \boldsymbol{b}_0$ and $\boldsymbol{B}(1) = \boldsymbol{b}_1$

Endpoint Tangent Property: $\boldsymbol{B}'(0) = n(\boldsymbol{b}_1 - \boldsymbol{b}_0)$ and $\boldsymbol{B}'(1) = n(\boldsymbol{b}_n - \boldsymbol{b}_{n-1})$

Section 5 The de Casteljau Algorithm

The de Casteljau algorithm gives us a simple method of evaluating a point on a Bézier curve for a given $t \in [0,1]$. Geometrically, it works by finding points along each side of the control polygon, where each point is length along the side is proportional to t. Using these points, it creates a new polygon which has one less side than the previous. The process is repeated n times until we obtain a single point which is along the curve. We will first define and prove the general algorithm, then we will explore some examples with the cubic Bézier curve.

Section 5.1 Defining the de Casteljau Algorithm

Theorem 5.1. The de Casteljau algorithm

Let a Bézier curve, $\boldsymbol{B}(t)$, of degree n be defined by control points $\boldsymbol{b}_0, \boldsymbol{b}_1, \dots, \boldsymbol{b}_n$ and $t \in [0,1]$. Then $\boldsymbol{B}(t)$=$\boldsymbol{b}_0^n$, where

$$\begin{cases} \boldsymbol{b}_0^i = \boldsymbol{b}_i \\ \boldsymbol{b}_i^j = (1-t)\boldsymbol{b}_i^{j-1} + t\boldsymbol{b}_{i+1}^{j-1} \end{cases}$$

For $j = 1, \dots, n$, and $i = 0, \dots, n-j$.

Proof. Recall the recursion property of the Bernstein polynomials, that is,

$$B_i^n(t) = (1-t)B_i^{n-1}(t) + tB_{i-1}^{n-1}(t)$$

So

$$\boldsymbol{B}(t) = \sum_{i=0}^{n} \boldsymbol{b}_i B_i^n(t) = \sum_{i=0}^{n} \boldsymbol{b}_i((1-t)B_i^{n-1}(t) + tB_{i-1}^{n-1}(t))$$

$$= \sum_{i=0}^{n} \boldsymbol{b}_i (1-t) B_i^{n-1}(t) + \sum_{i=0}^{n} \boldsymbol{b}_i t B_{i-1}^{n-1}(t)$$

Since $B_n^{n-1}(t) = B_{-1}^{n-1}(t) = 0$,

$$\boldsymbol{B}(t) = \sum_{i=0}^{n-1} \boldsymbol{b}_i (1-t) B_i^{n-1}(t) + \sum_{i=1}^{n} \boldsymbol{b}_i t B_{i-1}^{n-1}(t)$$

Now we modify the second summation by replacing i with $i+1$

$$\boldsymbol{B}(t) = \sum_{i=0}^{n-1} \boldsymbol{b}_i (1-t) B_i^{n-1}(t) + \sum_{i=0}^{n-1} \boldsymbol{b}_{i+1} t B_i^{n-1}(t)$$

$$= \sum_{i=0}^{n-1} (\boldsymbol{b}_i (1-t) + \boldsymbol{b}_{i+1} t)\, B_i^{n-1}(t)$$

Setting $\boldsymbol{b}_i^1 = \boldsymbol{b}_i(1-t) + \boldsymbol{b}_{i+1} t = \boldsymbol{b}_i^0(1-t) + \boldsymbol{b}_{i+1}^0 t$ for $i = 0, \dots, n-1$, we obtain

$$\boldsymbol{B}(t) = \sum_{i=0}^{n-1} \boldsymbol{b}_i^1 B_i^{n-1}(t)$$

Which gives $\boldsymbol{B}(t)$ as a Bézier curve of degree $n-1$ with control points $\boldsymbol{b}_0^1, \boldsymbol{b}_1^1, \dots, \boldsymbol{b}_{n-1}^1$.

Following this same logic, we see that

$$\boldsymbol{B}(t) = \sum_{i=0}^{n-2} \boldsymbol{b}_i^2 B_i^{n-2}(t)$$

where $\boldsymbol{b}_{i+1}^2 = \boldsymbol{b}_i^1(1-t) + \boldsymbol{b}_{i+1}^1 t$ for $i = 0, \dots, n-2$. So again, following the same argument, we see that in general

$$\boldsymbol{B}(t) = \sum_{i=0}^{n-j} \boldsymbol{b}_i^j B_i^{n-j}(t)$$

Where $\boldsymbol{b}_i^j = \boldsymbol{b}_i^{j-1}(1-t) + \boldsymbol{b}_{i+1}^{j-i}t$ for $i = 0, \dots, n-j$. In the case where $j = n$, we see that

$$\boldsymbol{B}(t) = \sum_{i=0}^{0} \boldsymbol{b}_i^n B_i^{n-n}(t) = \boldsymbol{b}_0^n$$

[3, pg. 152]■

The de Casteljau algorithm can be visualized as a triangular set of values where $\boldsymbol{B}(t) = \boldsymbol{b}_0^n$ for a given t. The following is a triangular set that corresponds to a cubic Bézier curve. We will show how these are computed in example 5.1.

$$\begin{array}{llll} \boldsymbol{b}_0^0 & \boldsymbol{b}_1^0 & \boldsymbol{b}_2^0 & \boldsymbol{b}_3^0 \\ \boldsymbol{b}_0^1 & \boldsymbol{b}_1^1 & \boldsymbol{b}_2^1 & \\ \boldsymbol{b}_0^2 & \boldsymbol{b}_1^2 & & \\ \boldsymbol{b}_0^3 & & & \end{array}$$

Example 5.1

Let's consider the cubic Bézier curve from example 2.3 that is defined by the control points $(2,3), (-2,6), (-5,4)$ and $(-6,0)$. We are interested in finding the point along the curve $\boldsymbol{B}(0.25)$. We will build up the terms of the formula and work our way to $\boldsymbol{b}_0^3$. The top row of the triangle, that is, the terms $\boldsymbol{b}_i^0, i = 0 \dots 3$, are simply our control points that form the foundation of the formula. We begin by computing the 2nd row down on the triangle, that is, $\boldsymbol{b}_i^1, i = 0 \dots 2$.

$$\boldsymbol{b}_0^1 = .75\begin{bmatrix}2\\3\end{bmatrix} + .25\begin{bmatrix}-2\\6\end{bmatrix} = \begin{bmatrix}1\\3.75\end{bmatrix}$$

$$\boldsymbol{b}_1^1 = .75\begin{bmatrix}-2\\6\end{bmatrix} + .25\begin{bmatrix}-5\\4\end{bmatrix} = \begin{bmatrix}-2.75\\5.5\end{bmatrix}$$

$$\boldsymbol{b}_2^1 = .75\begin{bmatrix}-5\\4\end{bmatrix} + .25\begin{bmatrix}-6\\0\end{bmatrix} = \begin{bmatrix}-5.25\\3\end{bmatrix}$$

Now we use these results for our computations as we move on to the 3rd row down.

$$\boldsymbol{b}_0^2 = .75\begin{bmatrix}1\\3.75\end{bmatrix} + .25\begin{bmatrix}-2.75\\5.5\end{bmatrix} = \begin{bmatrix}0.0625\\4.1875\end{bmatrix}$$

$$\boldsymbol{b}_1^2 = .75\begin{bmatrix}-2.75\\5.5\end{bmatrix} + .25\begin{bmatrix}-5.25\\3\end{bmatrix} = \begin{bmatrix}-3.375\\4.875\end{bmatrix}$$

Again, we use these for our computations in the final row.

$$\boldsymbol{b}_0^3 = .75\begin{bmatrix}0.0625\\4.1875\end{bmatrix} + .25\begin{bmatrix}-3.375\\4.875\end{bmatrix} = \begin{bmatrix}-0.796875\\4.35938\end{bmatrix} = \boldsymbol{B}(0.25)$$

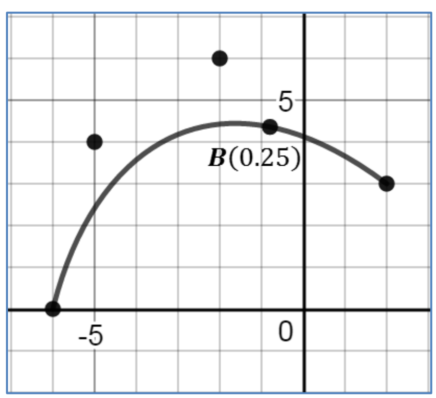

Figure 5.1. $\boldsymbol{B}(0.25)$ calculated using the de Casteljau algorithm.

Section 5.2 Subdivision of a Bézier curve

A Bézier curve is defined over the interval $[0, 1]$, but there are times when we may wish to deal with a portion of the curve. A common operation in vector graphics is to split a curve at a point, allowing one portion to remain the same while the other is manipulated. We can choose some $t = \alpha$ along the curve to split it into two curves such that the left portion of the curve is denoted $\boldsymbol{B}_{left}(t)$, and the right portion is $\boldsymbol{B}_{right}(t)$ where $\boldsymbol{B}_{left}(t)$ is defined over the interval $[0, \alpha]$, and $\boldsymbol{B}_{right}(t)$ is defined over $[\alpha, 1]$. Since $\boldsymbol{B}_{right}(t)$ and $\boldsymbol{B}_{left}(t)$ are polynomial curves, they can both individually be described by some Bézier curve on the interval $[0, 1]$.

Theorem 5.2. Subdividing a Bézier curve.

For a general a Bézier curve $\boldsymbol{B}(t) = \sum_{i=0}^{n} \boldsymbol{b}_i B_i^n(t)$, the control points of the two curve segments obtained by subdivision at a parameter value t are $\boldsymbol{b}_0^0, \boldsymbol{b}_0^1, \dots, \boldsymbol{b}_0^{n-1}, \boldsymbol{b}_0^n$ for $\boldsymbol{B}_{left}(t)$ and $\boldsymbol{b}_0^n, \boldsymbol{b}_1^{n-1}, \dots, \boldsymbol{b}_{n-1}^1, \boldsymbol{b}_n^0$ for $\boldsymbol{B}_{right}(t)$, where $\boldsymbol{b}_i^j$ are the points computed in the de Casteljau algorithm.

Proof. Let $\boldsymbol{B}(t)$ be a Bézier curve of degree n. First note that

$$\boldsymbol{b}_0^n = \boldsymbol{B}(t) = \sum_{i=0}^{n} \boldsymbol{b}_i B_i^n(t)$$

So, through this relationship, we then have the formula for any $\boldsymbol{b}_0^k$, $0 \le k \le n$, that is, any term along the leftmost column of the triangular set, which is

$$\boldsymbol{b}_0^k = \sum_{i=0}^{k} \boldsymbol{b}_i B_i^k(t)$$

which can be interpreted as the Bézier curve defined by the first k control points of $\boldsymbol{B}(t)$. Now consider the Bézier curve defined by the points along the leftmost column of the triangular set, that is, $\{\boldsymbol{b}_0^0, \boldsymbol{b}_0^1, \dots, \boldsymbol{b}_0^{n-1}, \boldsymbol{b}_0^n\}$. We denote this curve $\boldsymbol{B}'(t)$. Then

$$\boldsymbol{B}'(u) = \sum_{k=0}^{n} \boldsymbol{b}_0^{\boldsymbol{k}} B_k^n(u)$$

We use u as the parameter to avoid confusion as the domain of this curve is $[1, t]$ for an arbitrary but fixed t. Substituting our previous formula for $\boldsymbol{b}_0^{\boldsymbol{k}}$, we obtain

$$\boldsymbol{B}'(u) = \sum_{k=0}^{n} \left[\sum_{i=0}^{k} \boldsymbol{b}_i B_i^k(t) \right] B_k^n(u)$$

Note that $\boldsymbol{B}'(t)$ is still defined by the original set of control points in $\boldsymbol{B}(t)$. Now we will examine the coefficient of $\boldsymbol{b}_i$. Consider the coefficient, we will call a, of $\boldsymbol{b}_h$ for some $h \leq k$.

$$a = B_h^n(u)B_h^h(t) + B_{h+1}^n(u)B_h^{h+1}(t) + \cdots + B_n^n(u)B_h^n(t)$$

$$= \sum_{j=0}^{n-h} B_{h+j}^n(u) B_h^{h+j}(t)$$

$$= \sum_{j=0}^{n-h} \left[\frac{n!}{(h+j)!\big(n-(h+j)\big)!} u^{h+j}(1-u)^{n-(h+j)} \right] \left[\frac{(h+j)!}{h!\,j!} t^h (1-t)^j \right]$$

$$= \frac{n!}{h!}(tu)^h \sum_{j=0}^{n-h} \frac{1}{\big((n-h)-j\big)!\,j!} [u(1-t)]^j \big[(1-u)^{(n-h)-j}\big]$$

Multiplying the summation by $(n-h)!$ to put it in the binomial expansion form,

$$= \frac{n!}{h!\,(n-h)!}(tu)^h \sum_{j=0}^{n-h} \frac{(n-h)!}{\big((n-h)-j\big)!\,j!}[u(1-t)]^j\big[(1-u)^{(n-h)-j}\big]$$

Using the binomial expansion formula, we get

$$\sum_{j=0}^{n-h} \frac{(n-h)!}{\big((n-h)-j\big)!\,j!}[u(1-t)]^j\big[(1-u)^{(n-h)-j}\big] = [u(1-t)+(1-u)]^{n-h}$$

$$= (1-(tu))^{n-h}$$

Therefore, the coefficient a is

$$a = \frac{n!}{h!\,(n-h)!}(tu)^h(1-(tu))^{n-h} = B_h^n(tu)$$

Therefore, we see that

$$\boldsymbol{B}'(u) = \sum_{h=0}^{n} \boldsymbol{b}_h B_h^n(tu) = \boldsymbol{B}(tu)$$

So as u varies from 0 to 1, tu varies from 0 to t. Thus, $\boldsymbol{B}'(u)$ on the interval $[0,1]$ describes the same curve as $\boldsymbol{B}(t)$ on the interval $[0,t]$. The same procedure can be used to show that the points $\boldsymbol{b}_0^n, \boldsymbol{b}_1^{n-1}, \dots, \boldsymbol{b}_{n-1}^1, \boldsymbol{b}_n^0$ are the control points for a Bézier curve that is equivalent to $\boldsymbol{B}(t)$ on the interval $[0,t]$.

■

Section 6 Further Reading

Much of the literature on Bézier curves is written for and by graphics programmers. Naturally, these works are focused on application rather than theory. It was an interesting journey sifting through several works trying to extract enough content to provide sufficient mathematical depth for this paper. This paper, having gone so far in depth of the study of the Bézier curves, risks causing the reader to miss their simplistic beauty, especially in their modern applications. It is highly encouraged that if the reader is interested in this topic, that they explore further works that are more tailored to Bézier curves' graphical applications so they can appreciate the beauty of the simplicity of the curve.

References

[1] G. Timmerman, "Approximating Continuous Functions and Curves using Bernstein Polynomials," 2014.

[2] H. Prautzsch, W. Boehm and M. Paluszny, Mathematics and Visualization.

[3] D. Marsh, "Applied Geometry for Computer Graphics and CAD".

[4] K. I. Joy, "Affine Combinations, Barycentric Coordinates and Convex Combinations," University of California, Davis.

[5] S. Bernstein, "Demonstration of a theorem of Weierstrass based on the calculus of probabilities," 1913.